Solutions

1. Compute the real and imaginary parts of the following functions. Then, using the Cauchy–Riemann equations, proceed to show that they are holomorphic over their domain of definition and calculate their (complex) derivative.

(a) $f(z) = e^z$ over $\mathbb{C}$. Writing $z = x + iy$, we have:

$$f(z) = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Identifying the real and imaginary parts:

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y.$$

Checking the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial x} = e^x \sin y,$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y.$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, the function is holomorphic. The complex derivative is:

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = e^x \cos y + e^x \sin y i = e^z.$$

(b) $f(z) = z^3$ over $\mathbb{C}$.
Writing $z = x + iy$,

$$f(z) = (x + iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3.$$

Identifying the real and imaginary parts:

$$u(x, y) = x^3 - 3xy^2, \quad v(x, y) = 3x^2y - y^3.$$

Checking the Cauchy–Riemann equations confirms that f is holomorphic. The complex derivative is:

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = 3z^2.$$

(c) $f(z) = \frac{1}{z^2}$ over $\mathbb{C}^*$.
Writing $z = x + iy$, we express $f(z)$ as:

$$f(z) = \frac{1}{(x + iy)^2} = \frac{1}{x^2 - y^2 + 2ixy}.$$

Multiplying the numerator and denominator by the complex conjugate of the denominator:

$$f(z) = \frac{x^2 - y^2 - 2ixy}{(x^2 - y^2)^2 + 4x^2y^2}.$$

Identifying the real and imaginary parts:

$$u(x, y) = \frac{x^2 - y^2}{(x^2 - y^2)^2 + 4x^2y^2},$$
$$v(x, y) = \frac{-2xy}{(x^2 - y^2)^2 + 4x^2y^2}.$$

Checking the Cauchy–Riemann equations, we confirm that f is holomorphic. Finally, computing the derivative: $f'(z) = -\frac{2}{z^3}$.

2. Show that the following functions are entire and calculate their derivative.

(a) $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$
Note that e^{iz} and e^{-iz} are entire functions, as compositions of the entire function e^z with the entire function $\pm iz$, so their sum is also entire. Differentiating, using the chain rule for the derivative of the composition, we have:

$$\cos'(z) = \frac{(e^{iz})' + (e^{-iz})'}{2} = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z).$$

(b) $\cosh(z) = \frac{e^z+e^{-z}}{2}$
Entire with analogous justification as above. Differentiating:

$$\cosh'(z) = \frac{(e^z)' + (e^{-z})'}{2} = \frac{e^z - e^{-z}}{2} = \sinh(z).$$

(c) $\sinh(z) = \frac{e^z-e^{-z}}{2}$
Entire with analogous justification as above. Differentiating:

$$\sinh'(z) = \cosh(z).$$

3. The function $f(z) = (\text{Re}(z))^2$ is not holomorphic because its imaginary part is zero and does not satisfy the Cauchy–Riemann equations.

Writing $z = x + iy$, we have: $f(x + yi) = x^2$, so

$$u(x, y) = x^2, \quad v(x, y) = 0.$$

Computing the partial derivatives:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 0$$

and

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.$$

The Cauchy–Riemann equations require: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Substituting, we find:

$$2x = 0, \quad 0 = 0.$$

The second equation holds, but the first requires $x = 0$, so it does not hold on all of $\mathbb{C}$ (i.e. the domain of f; in fact, this means that there is no open set on which the Cauchy–Riemann equations hold for f). Therefore, $f(z)$ is not holomorphic.

4. Given that $f : \Omega \to \mathbb{C}$ is holomorphic with real and imaginary parts $u(x, y)$ and $v(x, y)$, we show that u and v satisfy the Laplace equation, meaning they are harmonic functions. Note that, as we have mentioned in class, a holomorphic function in fact has to be infinitely many times differentiable, so there is no issue with differentiating the expressions below and commuting derivatives.

By definition of a holomorphic function, u and v satisfy the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We now compute the second derivatives of u:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}.$$

Similarly, differentiating $\frac{\partial u}{\partial y}$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial x \partial y}.$$

Adding these two equations, we obtain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

Thus, u satisfies the Laplace equation, $\Delta u = 0$. By the same procedure, we differentiate v and obtain: $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. Therefore, v is also harmonic, completing the proof.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. We express the Cauchy–Riemann equations in polar coordinates (r, θ), where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

The standard Cauchy–Riemann equations in Cartesian coordinates are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

Using the chain rule, we can relate the coordinate derivatives in the (x, y) coordinate system to those in the (r, θ) coordinate system:

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin(\theta)\frac{\partial}{\partial x} + r\cos(\theta)\frac{\partial}{\partial y}$$

so, solving for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$:

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta},$$
$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}.$$

Applying these transformations to the Cauchy–Riemann equations (1):

$$\cos(\theta)\frac{\partial u}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial u}{\partial \theta} = \sin(\theta)\frac{\partial v}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial v}{\partial \theta},$$
$$\sin(\theta)\frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial u}{\partial \theta} = -\cos(\theta)\frac{\partial v}{\partial r} + \frac{\sin(\theta)}{r}\frac{\partial v}{\partial \theta}.$$

Multiplying the first equation by $\cos(\theta)$ and the second by $\sin(\theta)$, then summing, we obtain:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Similarly, multiplying the first equation by $\sin(\theta)$ and the second by $-\cos(\theta)$, then summing, we get:

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

These are the Cauchy–Riemann equations in polar coordinates.

6. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function given by:

$$f(x + iy) = u(x, y) + iv(x, y),$$

where the real part is given as:

$$u(x, y) = e^{(x^2-y^2)}\cos(2xy).$$

We need to find $v(x, y)$.

Since $f(x + iy)$ is entire, it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

First, we compute the necessary derivatives of $u(x, y)$:

$$\frac{\partial u}{\partial x} = e^{(x^2-y^2)}\left[2x\cos(2xy) - 2y\sin(2xy)\right],$$
$$\frac{\partial u}{\partial y} = e^{(x^2-y^2)}\left[-2y\cos(2xy) - 2x\sin(2xy)\right].$$

Using the first Cauchy-Riemann equation:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{(x^2-y^2)}\left[2x\cos(2xy) - 2y\sin(2xy)\right].$$

Integrating both sides with respect to y:

$$\begin{aligned} v(x,y) &= \int e^{(x^2-y^2)}\left[2x\cos(2xy) - 2y\sin(2xy)\right]dy + C(x) \\ &= \int e^{(x^2-y^2)}\left[\frac{\partial}{\partial y}(\sin(2xy)) - 2y\sin(2xy)\right]dy + C(x) \\ &= \int e^{(x^2-y^2)}\frac{\partial}{\partial y}(\sin(2xy))dy - \int e^{(x^2-y^2)}2y\sin(2xy)dy + C(x), \end{aligned}$$

where $C(x)$ is an arbitrary function of x. Integrating by parts indside the first integral, we get

$$\begin{aligned} v(x,y) =& e^{(x^2-y^2)}\sin(2xy) - \int \frac{\partial}{\partial y}\big(e^{(x^2-y^2)}\big)\sin(2xy)dy - \int e^{(x^2-y^2)}2y\sin(2xy)dy + C(x) \\ =& e^{(x^2-y^2)}\sin(2xy) + \int 2y\big(e^{(x^2-y^2)}\big)\sin(2xy)dy - \int e^{(x^2-y^2)}2y\sin(2xy)dy + C(x). \end{aligned}$$

Note that the two integrals cancel out, thus:

$$v(x,y) = e^{(x^2-y^2)}\sin(2xy) + C(x). \tag{2}$$

To determine $C(x)$, we use the second Cauchy-Riemann equation:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{3}$$

Differentiating $v(x, y)$ as given by (2):

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(e^{(x^2-y^2)}\sin(2xy) + C(x)\right)$$

$$=2xe^{(x^2-y^2)}\sin(2xy) + e^{(x^2-y^2)} \cdot 2y\cos(2xy) + C'(x)$$
$$=e^{(x^2-y^2)}\left[2x\sin(2xy) + 2y\cos(2xy)\right] + C'(x).$$

In view of (3), this must be equal to $-\frac{\partial u}{\partial y}$:

$$e^{(x^2-y^2)}\left[2x\sin(2xy) + 2y\cos(2xy)\right] + C'(x) = e^{(x^2-y^2)}\left[2x\sin(2xy) + 2y\cos(2xy)\right].$$

Canceling the common terms:

$$C'(x) = 0 \Rightarrow C(x) = \text{constant}.$$

So the final expression for $v(x,y)$ is:

$$v(x,y) = e^{(x^2-y^2)}\sin(2xy) + C,$$

where C is an arbitrary real constant.

Thus, the function $f(x+iy)$ is:

$$f(x+iy) = e^{(x^2-y^2)}\left[\cos(2xy) + i\sin(2xy)\right] + C,$$

or

$$f(z) = e^{z^2} + C$$

(using the fact that $\cos(2xy) + i\sin(2xy) = e^{i(2xy)}$).